SERIES APPROXIMATE SOLUTION METHOD FOR A BOUNDARY LAYER PROBLEM IN UNBOUNDED DOMAIN

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Abstract:

In this paper, we present a series approximate solution method for a boundary layer problem in unbounded domain by employing a reliable combination of modified iterative decomposition method (MIDM) and diagonal padé approximants. Boundary condition at infinity, poses a major problem generally to most numerical solution techniques. Without using the pade approximation, the semi analytic solution obtained failed to satisfy the boundary conditions at infinity. The proposed scheme finds the solution without discretization or restrictive assumptions, free from round off errors and so reduces computational burden to a great extent. Comparison is made between the obtained results and other semi-analytical methods mentioned in literature. The MIDM-Pade provides a simple, lesser iteration and effective technique for the solution of boundary layer problem in unbounded domain devoid of use of special polynomial. The results obtained attest to these assertions by being in excellent agreement with series solutions of other methods in literature.

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Key Words: Approximation, Boundary layer problem, Differential equation, Efficiency, Modified iterative decomposition method, Pade' approximants.

Introduction

The decomposition method began in 1980 as postulated by George Adomian (1994) and has been applied to a wide class of functional differential, integral and integro-differential equations. Various modifications of Adomian decomposition has been carried out to further enhance its suitability, efficiency and provision of accurate and easily computable series solutions for many classes of linear and nonlinear differential equations (Adomian 1994, Khan and Faraz 2011). Noor and Moyud-din (2009) presented a modified variational iteration method for solving boundary layer problem in unbounded domain that is based on coupled standard variational iteration method and He's polynomials. Other forms of Adomian's methods were proposed by Dafturdar-Gejji and Jafari (2006), Taiwo et al. (2009, 2018), Osilagun and Taiwo (2021) have been effectively used to solve a large class of linear and nonlinear equations. The Laplace transform decomposition algorithm is another approach based on Adomian's method that was introduced by Khuri (2001) and Khan (2009). It involves the use of Laplace transform to replace differential operators with simple algebraic operation on the transform. This method approximates the exact solution with a high degree of accuracy using only a few terms. Recently, several techniques including Adomian decomposition method (ADM), variational iteration method (VIM), finite difference method (FDM), differential transform method (DTM), and polynomial spline and homotopy perturbation methods (HPM) have been developed for solving boundary layer problems. Most of these methods have their inbuilt deficiencies such as the calculation of special Adomian polynomials, identification of Lagrange multiplier, divergent results, huge computational burden, inability of these semi-analytic methods to satisfy boundary conditions at infinity, Wazwaz (2006), Xu (2007), Noor and Mohyud-din (2008, 2009), Hussein and Khan (2010), Rashidi (2010.), Khan and Faraz (2011), Peker et al. (2011). The modified iterative decomposition method is combined with the diagonal padé approximants to obtain the approximate solution of a boundary layer problem in unbounded domain which arises in fluid mechanics that is devoid of use of special polynomials filled with complexity. In the earlier works of Wazwaz (1997) and Kuiken (1981), the behaviours of the infinite series solution was of great concern. Boyd (1997) and Baker (1975) have shown that power series in isolation may not be useful to boundary value problems. This is as a result of the possibility that the radius of convergence may not be sufficiently large to contain the boundaries of the domain. This justifies the combination of the modified iterative decomposition method or any series solution with the diagonal padé approximants as an effective tool to handle boundary value problem in infinite or semi-definite domains.

Analysis of the Methodology

Description of modified iterative decomposition Method (MIDM)-PADE' approximants.

In order to elucidate succinctness of the proposed modified iterative method's solution technique, we considered the general form of a second order, nonlinear, non homogeneous differential equation with initial conditions as given below

$$u''(x)+ p(x)u'(x)+q(x)u(x)+N[u(y)] = g(x) , \tag{1}$$

$$u(x) = \alpha, \qquad u'(x) = \beta. \tag{2}$$

Equation (1) can be put in operator form as

$$Lu + Ru + Nu = g(x) \tag{3}$$

where the operator $L = \dfrac{d^2(.)}{dx^2}$, because L is invertible, L^{-1} exists and it is a two-fold definite integral defined by

$L^{-1}(.) = \int_0^x \int_0^x (.)\, dsds$.

Applying L^{-1} to both sides of equation (3) and using the initial conditions, yield

$$u(x) = f(x) - L^{-1}[N(u)] \tag{4}$$

where

$$f(x) = \alpha + \beta(x) + L^{-1}[g(x) - Ru(x)] \tag{5}$$

The nonlinear operator $N(u)$ is decomposed as follow

$$N\left(\sum_{n=0}^{\infty} u_n\right) = N(u_0) + \sum_{n=0}^{\infty}\left\{N\left(\sum_{j=0}^{\infty} u_j\right) - N\left(\sum_{j=0}^{n-1} u_j\right)\right\} \tag{6}$$

On further simplification, we obtain the recursive relation

$$\begin{aligned} u_0 &= f(x) \\ u_1 &= L^{-1}[N(u_0)] \\ u_{n+1} &= L^{-1}\left[N\left(\sum_{i=1}^{n} u_i\right) - L^{-1} N\left(\sum_{i=1}^{n-1} u_i\right)\right], \; n \ge 1 \end{aligned} \tag{7}$$

where $f(x)$ represent the term arising from the source term and the prescribed initial conditions. The proposed technique is based on the assumption that the zeroth component $f(x)$ can be divided into two equal parts, such that

$$f(x) = f_0(x) + f_1(x) \tag{8}$$

The technique assigns only the part of $f_0(x)$ to the zeroth component of u_0 while the remaining part of $f_1(x)$ is combined only with other terms in u_1 of equation (7). Moreover, the current trend in numerical solutions of differential equations is towards efficiency, simple algorithm devoid of complexity and ensuring high level of accuracy. Thus, a modified recursive algorithm is obtained below by

$$\begin{aligned} u_0 &= f_0(x) \\ u_1 &= f_1(x) + L^{-1}[N(u_0)] \\ u_{n+1} &= L^{-1}\left[N\left(\sum_{i=1}^{n} u_i\right) - L^{-1} N\left(\sum_{i=1}^{n-1} u_i\right)\right], \; n \ge 1 \end{aligned} \tag{9}$$

The success of recursive scheme in (9) depends on the proper selection of $f_0(x)$ as the initial solution to avoid noise oscillation during the iteration process. Although, the choice of $f_0(x)$ is based on trial criteria; yet the proposed algorithm reduces computational burden when compared to the iterative method. It may also give exact solution with two or fewer iterations only without necessarily using special polynomials such as Adomian polynomials, Bell's polynomials and He's polynomials.

Padé Approximants

Padé approximant is the best approximant of a function with a rational function of a certain order. By this technique, the approximant power series agrees with the truncated series it is approximating and further extends the range of validity of the initial polynomial as an added advantage. According to Baker (1975), Pade' approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function $f(x)$. The [M|K] Pade' approximants to a function $f(x)$ is defined by

$$f(x)=\sum_{i=0}^{n} c_i x^i = \frac{\sum_{I=0}^{m} a_i x^i}{1+\sum_{i=1}^{k} b_i x^i} \qquad (10)$$

in the neighborhood of the origin such that there is no common factor between the numerator and the denominator, where $b_0 = 1$

So, $\sum_{i=0}^{n} c_i x^i (1+\sum_{i=0}^{k} b_i x^i) - \sum_{i=0}^{m} a_i x^i = O(x^{m+k+1})$ (11)

We conclude this section by the construction of a diagonal Pade' approximants [M/M]

Suppose, that $f(x)$) has a Taylor Series given by $f(x)=\sum_{i=0}^{n} c_i x^i$

$$f(x)=\frac{\sum_{i=0}^{m} a_i x^i}{1+\sum_{i=1}^{m} b_i x^i} = c_0 + c_1 x + c_2 x^2 \cdots + c_{2m} x^{2m}$$

$$\Rightarrow \sum_{i=0}^{m} a_i x^i = \left(1+\sum_{i=1}^{m} b_i x^i\right)\left(\sum_{k=0}^{2m} c_k x^k\right)$$

$= c_0 + (c_1 + b_1 c_0)x + (c_2 + b_1 c_1 + b_2 c_0)x^2 + (c_3 + b_1 c_2 + b_2 c_1 + b_3 c_0)x^3 + \cdots$ (12)

Set,

$[c_0 + (c_1 + b_1 c_0)x + (c_2 + b_1 c_1 + b_2 c_0)x^2 + (c_3 + b_1 c_2 + b_2 c_1 + b_3 c_0)x^3 + \cdots - [a_0 + a_1 x + a_2 x^2 + \cdots] = \sum_{i=0}^{m} d_i x^i$ (13)

Putting $d_i = 0,\ i = 0, 1, 2, 3, \ldots$ and equating the power of x leads to a system of linear equations, that is solved by Gaussian elimination method to obtain the unknown coefficients

$$a_0, a_1, \ldots, a_m, \qquad b_1, b_2, \ldots, b_m$$

Which results to the required [M|M] approximant.

Finally, diagonal pade approximants of different orders such as [2|2], [4|4] or [6|6], can be obtained using the symbolic calculus software, Maple.

Application

In this section, we apply the modified iterative decomposition method combined with the diagonal padé approximants to a third order boundary layer problem and the results obtained are presented in tabular form for easy comparison with some of the different methods mentioned in literature. All symbolic computation is implemented using Maple solver.

Consider $f'''(x) + (n-1) f(x) f'(x) - 2n (f'(x))^2 = 0, \quad n \succ 0$ (14)

With boundary conditions

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0 \qquad (15))$$

Where $f''(0) = \alpha < 0$ (16)

By applying the recursive algorithm (9) and using the initial conditions, yield the following

$$f_0(x) = x$$

$$f_1(x) = \frac{x^2}{2}\alpha + \frac{nx^3}{3}$$

$$f_2(x) = \frac{\alpha(3n+1)}{24}x^4 + \left(\frac{n}{30} + \frac{n^2}{30} + \frac{n}{40}\alpha^2 + \frac{\alpha^2}{120}\right)x^5 + \left(\frac{n\alpha}{90} + \frac{n^2\alpha}{45}\right)x^6 + \left(\frac{n^2+2n^3}{315}\right)x^7$$

$$\begin{aligned} f_3(x) = {} & \frac{n^2}{315}x^7 + \frac{n\alpha}{42}x^6 + \frac{n^2\alpha}{240}x^6 + \frac{n^4}{6449625}x^{15} + \frac{16n^5}{19348875}x^{15} + \frac{4n^6}{2764125}x^{15} + \frac{n^4}{61425}x^{13} + \\ & \frac{179n^5}{8108100}x^{13} + \frac{31n^3}{8108100}x^{13} + \frac{n^3}{7700}x^{11} + \frac{4n^4}{17325}x^{11} + \frac{\alpha^4}{712800}x^{11} + \frac{n^2}{44550}x^{11} + \frac{\alpha^3}{64800}x^{10} + \\ & \frac{19n^2}{22680}x^9 + \frac{n^3}{567}x^9 + \frac{\alpha^2}{24192}x^9 + \frac{11\alpha^3}{40320}x^8 + \frac{11\alpha^2}{5040}x^7 + \frac{n}{315}x^7 + \frac{\alpha}{240}x^6 + \frac{16n^7 19348875}{}x^{15} + \frac{n^6}{103950}x^{13} \\ & + \frac{23n^5}{207900}x^{11} + \frac{13n^4}{22680}x^9 + \frac{829n^2\alpha^2}{362880}x^9 + \frac{59n\alpha^2}{72576}x^9 + \frac{19n\alpha}{6720}x^8 + \frac{n\alpha^3}{960}x^8 + \frac{n\alpha^2}{120}x^7 + \frac{557n\ \alpha^2}{362880}x^9 + \frac{3n^2\alpha^3}{4480} \\ & + \frac{3n^2\alpha^2}{560}x^7 + \frac{8n^6\alpha}{1289925}x^{14} + \frac{313n^5\alpha^2}{16216200}x^{13} + \frac{37n^5\alpha}{498960}x^{12} + \frac{7n^4\alpha^3}{237600}x^{12} + \frac{n^4\alpha^2}{4950}x^{11} + \frac{n^3\alpha^4}{52800}x^{11} + \frac{53n^4}{75600}x^{16} \\ & + \frac{n^3\alpha^3}{4800}x^{10} + \frac{17n^3\alpha}{4032}x^8 + \frac{8n^4\alpha}{1289925}x^{14} + \frac{2n^5\alpha}{184275}x^{14} + \frac{n^3\alpha}{859950}x^{14} + \frac{n^3\alpha^2}{56700}x^{13} + \frac{n^4\alpha^2}{30800}x^{13} + \frac{101n^4\alpha}{32432400}x^{12} \\ & + \frac{59n^3\alpha}{623700}x^{12} + \frac{247n^4\alpha}{1663200}x^{12} + \frac{13n^3\alpha^3}{285120}x^{12} + \frac{n^2\alpha^3}{44550}x^{12} + \frac{n\alpha^3}{285120}x^{12} + \frac{97n^2\alpha}{4989600}x^{12} + \frac{n^2\alpha^2}{5400}x^{11} + \frac{n\alpha^2}{32400}x^{11} \\ & + \frac{41n^3\alpha^2}{118800}x^{11} + \frac{n^2\alpha^4}{39600}x^{11} + \frac{n\alpha^4}{95040}x^{11} + \frac{n\alpha^3}{8640}x^{10} + \frac{n\alpha}{16200}x^{10} + \frac{151n^3\alpha}{113400}x^{10} + \frac{n^2\alpha^3}{3600}x^{10} + \frac{47n^2\alpha}{75600}x^{10} + \frac{73n^2\alpha}{10080}x^8 \end{aligned}$$

The series solution after three iterations is given by $f(x) = f_0 + f_1 + f_2 + f_3$

So,

$$\begin{aligned} f(x) = {} & x + \frac{\alpha}{2}x^2 + \frac{n}{3}x^3 + \left(\frac{\alpha}{24} + \frac{n\alpha}{8}\right)x^4 + \left(\frac{n^2}{30} + \frac{n\alpha^2}{40} + \frac{\alpha^2}{120} + \frac{n}{30}\right)x^5 + \left(\frac{19n^2\alpha}{720} + \frac{\alpha}{240} + \frac{n\alpha}{40}\right)x^6 \\ & + \left(\frac{n\alpha^2}{120} + \frac{n}{315} + \frac{2n^3}{315} + \frac{11\alpha^2}{540} + \frac{3n^2\alpha^2}{560} + \frac{2n^2}{315}\right)x^7 \\ & + \left(\frac{11\alpha^3}{40320} + \frac{33n^2\alpha 2}{4480} + \frac{23n\alpha}{5760} + \frac{\alpha}{2688} + \frac{167n^3\alpha}{40320} + \frac{n\alpha^3}{960}\right)x^8 \\ & + \left(\frac{n}{3780} + \frac{527n^3\alpha^2}{362880} + \frac{19n^3}{11340} + \frac{709n\alpha^2}{362880} + \frac{23n^2\alpha^2}{8064} + \frac{23n^2}{226780} + \frac{13n^4}{22680} + \frac{43\alpha^2}{120960}\right)x^9 + \ldots \end{aligned}$$

furthermore, we obtained the diagonal Padé approximants of different order $[2|2],[3|3],[4|4],[5|5]$ and $[6|6]$ of the solution $f(x)$ for an insight to the solution behavior.

Results and Discussion

Table 1: Comparison of the numerical values of $\alpha = f''(x) \quad for \ 0 < x < 1$ by using Padé approximation

N	Pade' approximants	MADM Wazwaz (2006)	MVIM Noor & Mohyud-din (2009)	MLDM Khan & Faraz (2011)	Proposed method
0.2	**[2|2]**	**-0.3872983347**	**-0.3872983347**	**-0.3872983347**	**-0.3872983347**
	[3|3]	**-0.3821533832**	**-0.3821533832**	**-0.3821533832**	**-0.3821533832**
	[4|4]	**-0.3819153845**	**-0.3819153845**	**-0.3819153845**	**-0.3819153845**
	[5|5]	**-0.3819148088**	**-0.3819148088**	**-0.3819148088**	**-0.3819148088**
	[6|6]	**-0.3819121854**	**-0.3819121854**	**-0.3819121854**	**-0.3819121854**
0.3	**[2|2]**	**-0.5773502692**	**0.5773502692**	**0.5773502692**	**0.5773502692**
	[3|3]	**-0.5615999244**	**-0.5615999244**	**-0.5615999244**	**-0.5615999244**
	[4|4]	**-0.5614066588**	**-0.5614066588**	**-0.5614066588**	**-0.5614066588**
	[5|5]	**-0.5614481405**	**-0.5614481405**	**-0.5614481405**	**-0.5614481405**
	[6|6]	**-0.5614491934**	**-0.5614491934**	**-0.5614491934**	**-0.5614491934**
0.4	**[2|2]**	**-0.6451506398**	**-0.6451506398**	**-0.6451506398**	**-0.6451506398**
	[3|3]	**-0.6391000575**	**-0.6391000575**	**-0.6391000575**	**-0.6391000575**
	[4|4]	**-0.6389732578**	**-0.6389732578**	**-0.6389732578**	**-0.6389732578**
	[5|5]	**-0.6389892681**	**-0.6389892681**	**-0.6389892681**	**-0.6389892681**
	[6|6]	**-0.6389734794**	**-0.638973479**	**-0.638973479**	**-0.638973479**
0.6	**[2|2]**	**-0.8407961591**	**-0.8407961591**	**-0.8407961591**	**-0.8407961591**
	[3|3]	**-0.8393603021**	**-0.8393603021**	**-0.8393603021**	**-0.8393603021**
	[4|4]	**-0.8396060478**	**-0.8396060478**	**-0.8396060478**	**-0.8396060478**
	[5|5]	**-0.8395875381**	**-0.8395875381**	**-0.8395875381**	**-0.8395875381**
	[6|6]	**-0.8396056769**	**-0.8396056769**	**-0.8396056769**	**-0.8396056769**
0.8	**[2|2]**	**-1.0079832070**	**-1.0079832070**	**-1.0079832070**	**-1.0079832070**
	[3|3]	**-1.0077969810**	**-1.0077969810**	**-1.0077969810**	**-1.0077969810**
	[4|4]	**-1.0076468280**	**-1.0076468280**	**-1.0076468280**	**-1.0076468280**
	[5|5]	**-1.0076468280**	**-1.0076468280**	**-1.0076468280**	**-1.0076468280**
	[6|6]	**-1.0077921000**	**-1.0077921000**	**-1.0077921000**	**-1.0077921000**

The series solution accurate and further understanding of the solution behavior is enhanced by the diagonal Pade approximations exhibited in Table 2 using the series solution $f(x)$ obtained earlier in section three.

Table 2: Comparison of the numerical value of $\alpha = f''(0)$, using diagonal Pade' approximants

N	MADM Wazwaz (2006)	HPM Xu (2007)	MVIM Noor & Mohyudin (2009)	HPM Xu (2007)	MLDM Khan & Faraz (2011)	Present Method
4	-2.483954032	-2.5568	-2.483954032	-2.5568	-2.483954032	-2.483954032
10	-4.026385103	-4.0476	-4.026385103	-4.0476	-4.026385103	-4.026385103
100	-12.84334315	-12.8501	-12.84334315	-12.8501	-12.84334315	-12.84334315
1000	-40.65538218	- 40.4556	-40.65538218	40.4556	-40.65538218	-40.65538218
5000	-104.8420672	- 90.9127	-104.8420672	- 90.9127	-104.8420672	-104.8420672

Conclusion

The modified iterative decomposition method combined with the diagonal Padé approximants for solving boundary layer problem in unbounded domain has been presented. The convergence of the technique is also exhibited as shown in Tables 1 and **2**. An analytic approach was used to obtain numerical values of $f''(x)$ for various values of n. The proposed method further attests to the fact that $f''(x)$ decays algebraically for $0 < n < 1$ and decays exponentially for $n > 1$ as x tends to infinity as earlier claimed in Kuiken (1981), Wazwaz (2006), Xu (2007), Noor & Mohyudin (2009) and Khan & Faraz (2011). Results of the proposed method compares favourably/excellently with the existing solutions in the literature. Moreover, it uses the least iteration, is simple in principles and convenient for computer algorithms. The proposed technique does not require any form of linearization, perturbation, discretization or restrictive assumption and is free from round off error.

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